Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Exercise sheet 10** (07.01.2026)

(10.1) Exercise: Rational functions on affine varieties.
Let $K \subseteq L$ be a field extension, where L is algebraically closed, let $\mathbf{V}$ be an irreducible affine variety, and let $K(\mathbf{V})$ be the field of rational functions on $\mathbf{V}$. Moreover, let $f, f', g, g' \in K[\mathbf{V}]$, where $gg' \neq 0$, and let $\emptyset \neq U \subseteq D_{gg'}$ open, such that $\frac{f(v)}{g(v)} = \frac{f'(v)}{g'(v)}$ for all $v \in U$. Show that $\frac{f}{g} = \frac{f'}{g'} \in K(\mathbf{V})$.

(10.2) Exercise: Rational functions.
Let $K \subseteq L$ be a field extension, where L is algebraically closed, and let V be an irreducible (quasi-projective) variety.

a) Show that any affine open subset $\emptyset \neq U \subseteq V$ is irreducible, and that $K(U)$ is independent of the choice of U. Hence $K(U)$ is (unambiguously) called the field of **rational functions** on V; we again denote it by $K(V)$. Describe $K(V)$ if V is quasi-affine.

b) Show that for any open subset $U \subseteq V$ the K-algebra $\mathcal{O}_V(U)$ can be identified with a K-subalgebra of $K(V)$, such that $\mathrm{Q}(\mathcal{O}_V(U)) = K(V)$. If $U' \subseteq U$ is open, describe the associated restriction map in terms of K-subalgebras of $K(V)$. If $U \subseteq V$ and $U' \subseteq V$ are open, show that $\mathcal{O}_V(U \cup U') = \mathcal{O}_V(U) \cap \mathcal{O}_V(U') \subseteq K(V)$.

c) Let $\varphi\colon V \to W$ be a morphism of irreducible varieties such that $\varphi(V)$ contains an open subset of W. (In particular φ is dominant.) Show that φ^* induces a field extension $K(W) \subseteq K(V)$. Can the assumption on $\varphi(V)$ be dispensed of?

(10.3) Exercise: Rational functions on projective varieties.
Let $K \subseteq L$ be a field extension, where L is algebraically closed, let $\mathbf{V} \subseteq \mathbf{P}$ be an irreducible projective variety, having (homogenuous prime) vanishing ideal $I \trianglelefteq A^\sharp$, and let $R := (A^\sharp_{A^\sharp \setminus I})_0 = \{\frac{f}{g} \in Q^\sharp(A^\sharp); g \notin I\} \subseteq Q^\sharp(A^\sharp)$ be the associated graded localisation of $A^\sharp$. Show that $IR \trianglelefteq R$ is the unique maximal homogenuous ideal of R, and that we have $K(\mathbf{V}) \cong R/IR$.

(10.4) Exercise: Morphisms of quasi-projective varieties.
Let $K \subseteq L$ be a field extension, where L is algebraically closed, and let $V \subseteq \mathbf{P}^n$ be a quasi-projective variety.

a) Let $f_0, \ldots, f_m \in A^\sharp$, where $m \in \mathbb{N}_0$, be homogeneous of the same degree, such that $\mathbf{V}_L(f_0, \ldots, f_m) \cap V = \emptyset$. Show that $\varphi\colon V \to \mathbf{P}^m \colon v \mapsto [f_0(v)\colon \ldots \colon f_m(v)]$ is a morphism of varieties. Moreover, if $g_0, \ldots, g_m \in A^\sharp$ also fulfill the above properties, show that $V \to \mathbf{P}^m \colon v \mapsto [g_0(v)\colon \ldots \colon g_m(v)]$ equals φ if and only if $f_i g_j|_V = f_j g_i|_V$, for all $i, j \in \{0, \ldots, m\}$.

b) Let $\psi\colon V \to \mathbf{P}^m$ be a morphism of varieties. Show that for any $v \in V$ there is an open neighborhood $v \in U \subseteq V$ such that $\psi|_U$ is of the above form.