Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Exercise sheet 9** (10.12.2025)

(9.1) Exercise: Duality.
Let K be a field, and let $\mathbf{vec}_K$ be the category of finite-dimensional K-vector spaces, having object set $\mathrm{Ob}(\mathbf{vec}_K) := \{K^n; n \in \mathbb{N}_0\}$, and having the set $\mathrm{Hom}_K(?, ??)$ of all K-linear maps as morphisms.

a) Show that $\mathbf{vec}_K$ is a category indeed. Moreover, show that it is **skeletal**, that is its objects are pairwise non-isomorphic.

b) Show that assigning $V \mapsto V^* := \mathrm{Hom}_K(V, K)$ and

$$\mathrm{Hom}_K(V, W) \to \mathrm{Hom}_K(W^*, V^*) \colon \varphi \mapsto (\varphi^* \colon \lambda \mapsto \lambda \circ \varphi)$$

defines a contravariant **duality (endo-)**functor $^* \colon \mathbf{vec}_K \to \mathbf{vec}_K$.

c) Show that the **biduality functor** ** is isomorphic to the identity functor.

(9.2) Exercise: Locally constant sheaves.
Let $\mathcal{V}$ be a topological space.

a) Given a presheaf $\mathcal{F}$ on $\mathcal{V}$, show that the following assertions are equivalent:

i) For any open subset $\emptyset \neq \mathcal{U} \subseteq \mathcal{V}$ the restriction map $\mathcal{F}(\mathcal{V}) \to \mathcal{F}(\mathcal{U})$ is bijective.

ii) $\mathcal{F}$ is a **constant sheaf**, that is there is a set A, equipped with the discrete topology, such that $\mathcal{F}(\mathcal{U})$ consists of the locally constant functions $\mathcal{U} \to A$.

iii) $\mathcal{F}$ is a **locally constant sheaf**, that is any point $v \in \mathcal{V}$ has an open neighborhood $\mathcal{U} \subseteq \mathcal{V}$ such that the restriction $\mathcal{F}|_{\mathcal{U}}$ is a constant sheaf.

b) Let $\mathcal{V}$ be connected, and let $\mathcal{F}$ be a constant sheaf on $\mathcal{V}$ such that $|\mathcal{F}(\mathcal{V})| \geq 2$. Show that $\mathcal{V}$ is irreducible.

c) Let $\mathcal{F}$ be a constant sheaf on $\mathcal{V}$, and let $\mathcal{F}_0$ be the **constant presheaf** on $\mathcal{V}$ with value $A := \mathcal{F}(\mathcal{V})$, that is $\mathcal{F}(\mathcal{U})$ consists of the constant functions $\mathcal{U} \to A$, for any open subset $\mathcal{U} \subseteq \mathcal{V}$. Show that $\mathcal{F}$ has the following universal property:

There exists a morphism of presheaves $\psi \colon \mathcal{F}_0 \to \mathcal{F}$, such that for any morphism of presheaves $\varphi_0 \colon \mathcal{F}_0 \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf, there is a unique morphism of sheaves $\varphi \colon \mathcal{F} \to \mathcal{G}$ such that $\varphi_0 = \psi \cdot \varphi$. ($\mathcal{F}$ is called the **sheafification** of $\mathcal{F}_0$.)

(9.3) Exercise: Direct image sheaves.
a) Let $\mathcal{V}$ be a topological space, and let $\mathcal{A}$ be a category. Show that $\mathbf{PSh}(\mathcal{V}, \mathcal{A})$, with objects consisting of the presheaves on $\mathcal{V}$ with values in $\mathcal{A}$, and having all morphism of presheaves as morphisms, is a category. Similarly, show that $\mathbf{Sh}(\mathcal{V}, \mathcal{A})$, with objects consisting of the sheaves on $\mathcal{V}$ with values in $\mathcal{A}$, and having all morphism of presheaves as morphisms, is a category.

b) Let $\mathcal{W}$ be a topological space, let $\varphi \colon \mathcal{V} \to \mathcal{W}$ be continuous, and let $\mathcal{F} \in \mathbf{PSh}(\mathcal{V}, \mathcal{A})$. Show that for any $U' \subseteq U \subseteq \mathcal{W}$ open letting

$$U \mapsto \mathcal{F}(\varphi^{-1}(U)) \quad \text{and} \quad \rho^{U}_{U' \subseteq \mathcal{W}} := \rho^{\varphi^{-1}(U)}_{\varphi^{-1}(U') \subseteq \mathcal{V}}$$

defines a presheaf $\varphi_* \mathcal{F} \in \mathbf{PSh}(\mathcal{W}, \mathcal{A})$, being called the **direct image** of $\mathcal{F}$ under φ. Show that this gives rise to a functor $\varphi_* \colon \mathbf{PSh}(\mathcal{V}, \mathcal{A}) \to \mathbf{PSh}(\mathcal{W}, \mathcal{A})$.

Moreover, if $\mathcal{F}$ is a sheaf, show that $\varphi_* \mathcal{F}$ is a sheaf as well. Conclude that this gives rise to a functor $\varphi_* \colon \mathbf{Sh}(\mathcal{V}, \mathcal{A}) \to \mathbf{Sh}(\mathcal{W}, \mathcal{A})$.