Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Lecture 22** (22.12.2025)

(22.1) Affine varieties. a) We keep the earlier setting, let L be algebraically closed, and let $\mathbf{V} \subseteq L^n$ be an affine variety.

Theorem. Let $f \in K[\mathbf{V}]$. Then we have $\mathcal{O}_{\mathbf{V}}(D_f) = K[\mathbf{V}]_f$.

Proof. We may assume that $f \neq 0$, and let $\varphi \in \mathcal{O}_{\mathbf{V}}(D_f)$. Then there are $f_i, g_i \in K[\mathbf{V}]$ for $i \in \{1, \ldots, r\}$, such that $\varphi|_{D_{f_i}} = \frac{g_i}{f_i}|_{D_{f_i}}$, where $D_f = \bigcup_{i=1}^r D_{f_i}$. Then we have $(f_i g_j - f_j g_i)|_{D_{f_i f_j}} = 0$, for all i, j, hence $f_i f_j (f_i g_j - f_j g_i) = 0 \in K[\mathbf{V}]$. Rewriting this as $f_i^2 (f_j g_j) - f_j^2 (f_i g_i) = 0$, and observing $\varphi|_{D_{f_i}} = \frac{f_i g_i}{f_i^2}|_{D_{f_i}}$, we may assume that $\varphi|_{D_{f_i}} = \frac{g_i}{f_i}|_{D_{f_i}}$, where $f_i g_j - f_j g_i = 0 \in K[\mathbf{V}]$.

From $\mathbf{V}_{\mathbf{V}}(f) = \mathbf{V} \setminus D_f = \bigcap_{i=1}^r (\mathbf{V} \setminus D_{f_i}) = \bigcap_{i=1}^r \mathbf{V}_{\mathbf{V}}(f_i) = \mathbf{V}_{\mathbf{V}}(f_1, \ldots, f_r)$ we get $f \in \mathbf{I}_{\mathbf{V}}(\mathbf{V}_{\mathbf{V}}(f_1, \ldots, f_r)) = \sqrt{\langle f_1, \ldots, f_r \rangle}$, hence there are $k \in \mathbb{N}$ and $h_i \in K[\mathbf{V}]$ such that $f^k = \sum_{i=1}^r h_i f_i$. This yields $f^k g_j = \sum_{i=1}^r h_i f_i g_j = \sum_{i=1}^r h_i g_i f_j = (\sum_{i=1}^r h_i g_i) f_j \in K[\mathbf{V}]$. Thus letting $h := \sum_{i=1}^r h_i f_i \in K[\mathbf{V}]$ we get $\frac{g_j}{f_j}|_{D_{f_j}} = (\frac{h}{f^k})|_{D_{f_j}}$. This implies that $\varphi = \frac{h}{f^k}|_{D_f}$.

Since $f \in \mathcal{O}_{\mathbf{V}}(D_f)$ is a unit, there is a K-algebra epimorphism $\alpha \colon K[\mathbf{V}]_f \to \mathcal{O}_{\mathbf{V}}(D_f)$. In order to show that α in injective, let $(\frac{g}{f^k})|_{D_f} = (\frac{h}{f^l})|_{D_f}$, where $g, h \in K[\mathbf{V}]$ and $k, l \in \mathbb{N}_0$. Then we have $(gf^l - hf^k)|_{D_f} = 0$, which implies that $(gf^l - hf^k) f = 0 \in K[\mathbf{V}]$. Thus we have $\frac{g}{f^k} = \frac{h}{f^l} \in K[\mathbf{V}]_f$. ♯

Corollary. We have $\Gamma(\mathcal{O}_{\mathbf{V}}) = K[\mathbf{V}]$.

Recall that the Zariski topology on $\mathbf{V}$ can be recovered from $K[\mathbf{V}]$. Moreover, a map $\varphi \colon \mathbf{V} \to L$ is regular in the earlier sense, that is $\varphi \in K[\mathbf{V}]$, if and only if it is a regular function on $\mathbf{V}$, that is $\varphi \in \mathcal{O}_{\mathbf{V}}(\mathbf{V}) = \Gamma(\mathcal{O}_{\mathbf{V}})$.

We observe that the K-algebra of regular functions on any principal open subset of $\mathbf{V}$ can be recovered from $K[\mathbf{V}]$ as well, thus by the sheaf properties this holds for any open subset of $\mathbf{V}$. In other words, the structure sheaf of any quasi-affine variety is determined by the coordinate algebra of an affine variety it is open in. In particular, this holds for affine varieties themselves, so that the present definition of affine varieties coincides with the earlier one.

b) Let $\mathbf{W} \subseteq L^m$ be an affine variety. Then a map $\varphi \colon \mathbf{V} \to \mathbf{W}$ is regular in the earlier sense if and only if $\varphi^*(K[\mathbf{W}]) \subseteq K[\mathbf{V}]$; in this case $\varphi^* \colon K[\mathbf{W}] \to K[\mathbf{V}]$ is a homomorphism of K-algebras. Actually, the earlier notion of regularity indeed coincides with the present definition of morphisms:

Corollary. The map φ is regular if and only if it is a morphism of varieties.

Proof. If φ is a morphism of varieties, that is φ is continuous and for any open subset $U \subseteq \mathbf{W}$ we have $\varphi_U^*(\mathcal{O}_{\mathbf{W}}(U)) \subseteq \mathcal{O}_{\mathbf{V}}(\varphi^{-1}(U))$, then for $U = \mathbf{W}$ we get $\varphi^*(K[\mathbf{W}]) = \varphi_{\mathbf{W}}^*(\mathcal{O}_{\mathbf{W}}(\mathbf{W})) \subseteq \mathcal{O}_{\mathbf{V}}(\varphi^{-1}(\mathbf{W})) = \mathcal{O}_{\mathbf{V}}(\mathbf{V}) = K[\mathbf{V}]$.

Conversely, if φ is regular, then it is continuous. It remains to show that $\varphi^*(\mathcal{O}_{\mathbf{W}}(U)) \subseteq \mathcal{O}_{\mathbf{V}}(\varphi^{-1}(U))$ for any open subset $U \subseteq \mathbf{W}$. Since the principal open subsets are a basis of the Zariski topology on U, by the sheaf properties it suffices to consider the principal open subsets $D_f \subseteq \mathbf{W}$, where $f \in K[\mathbf{W}]$:

We have $\varphi^{-1}(D_f) = \{v \in \mathbf{V}; f(\varphi(v)) \neq 0\} = D_{\varphi^*(f)} \subseteq \mathbf{V}$. Moreover, the homomorphism $\varphi^* \colon K[\mathbf{W}] \to K[\mathbf{V}]$ extends naturally to $\varphi_f^* \colon K[\mathbf{W}]_f \to K[\mathbf{V}]_{\varphi^*(f)}$. Thus we have $\varphi_{D_f}^*(\mathcal{O}_{\mathbf{W}}(D_f)) = \varphi_f^*(K[\mathbf{W}]_f) \subseteq K[\mathbf{V}]_{\varphi^*(f)} = \mathcal{O}_{\mathbf{V}}(D_{\varphi^*(f)})$. $\sharp$

In particular, for $\mathbf{W} = L$, we conclude that the set $\Gamma(\mathcal{O}_{\mathbf{V}}) = K[\mathbf{V}]$ of regular functions on $\mathbf{V}$ consists precisely of the morphisms of (affine) varieties $\mathbf{V} \to L$.

(22.2) Principal open subsets of affine varieties. We keep the above notation, let L be algebraically closed, and let $\mathbf{V} \subseteq L^n$ be closed. We show that any principal open subset of $\mathbf{V}$ is isomorphic to an affine variety:

Let $I := \mathbf{I}_K(\mathbf{V}) \trianglelefteq A$, and let T be an indeterminate. Then for $f \in K[\mathbf{V}] = A/I$ let $\widehat{I}_f := \langle I, fT - 1 \rangle \trianglelefteq A[T]$ (which is independent of the chosen coset representative for f). We consider the affine closed set

$$\widehat{\mathbf{V}}_f := \mathbf{V}_L(\widehat{I}_f) = \{[v, t] \in \mathbf{V} \times L; f(v) \cdot t = 1\} \subseteq L^n \times L = L^{n+1}.$$

In order to show $K[\widehat{\mathbf{V}}_f] = A[T]/\mathbf{I}_K(\widehat{\mathbf{V}}_f) = A[T]/\widehat{I}_f$, we show that $A[T]/\widehat{I}_f$ is reduced: From $A[T]/\langle I \rangle \cong (A/I)[T]$ we get $A[T]/\widehat{I}_f \cong (A/I)[T]/\langle fT - 1 \rangle = K[\mathbf{V}][T]/\langle fT - 1 \rangle \cong K[\mathbf{V}]_f$. Now, since $K[\mathbf{V}]$ is reduced, so is $K[\mathbf{V}]_f$. $\sharp$
