Algebraic Geometry (WS 2025)

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(23.1) Principal open subsets of affine varieties, cont. We keep the earlier setting.

Theorem. We have $\widehat{\mathbf{V}}_f \cong D_f$ as varieties.

Proof. Let $\widehat{\pi}\colon L^{n+1} \to L^n$ be the projection onto the first n coordinates; since $\widehat{\pi}$ is a regular map (in the earlier sense), it is continuous. Thus $\widehat{\pi}$ restricts to a continuous map $\pi\colon \widehat{\mathbf{V}}_f \to D_f$, which is bijective having inverse $\pi^{-1}\colon D_f \to \widehat{\mathbf{V}}_f\colon v \mapsto [v, \frac{1}{f(v)}]$. We proceed to show that π is a homeomorphism:

A basis of the Zariski topology on $\widehat{\mathbf{V}}_f$ is given by the principal open subsets. Letting $\frac{g}{f^k} \in K[\mathbf{V}]_f$, for some $g \in K[\mathbf{V}]$ and $k \in \mathbb{N}_0$, we get $\widehat{D}_{\frac{g}{f^k}} = \widehat{D}_g \subseteq \widehat{\mathbf{V}}_f$, where $\mathcal{O}_{\widehat{\mathbf{V}}_f}(\widehat{D}_g) = (K[\mathbf{V}]_f)_g$. For $\widehat{D}_h \subseteq \widehat{D}_g$, for $h \in K[\mathbf{V}]$, restriction is given by the natural map $(K[\mathbf{V}]_f)_g \to (K[\mathbf{V}]_f)_h$; note that $g \in (K[\mathbf{V}]_f)_h$ is a unit.

Similarly, a basis of the Zariski topology on D_f is given by the principal open subsets $D_{fg} = D_g \cap D_f \subseteq D_f$, for some $g \in K[\mathbf{V}]$, where $\mathcal{O}_{D_f}(D_{fg}) = \mathcal{O}_{\mathbf{V}}(D_{fg}) = K[\mathbf{V}]_{fg}$. For $D_{fh} \subseteq D_{fg}$, where $h \in K[\mathbf{V}]$, restriction is given by the natural map $K[\mathbf{V}]_{fg} \to K[\mathbf{V}]_{fh}$; note that $fg \in K[\mathbf{V}]_{fh}$ is a unit.

From $g([v,t]) = g(v)$, for all $[v,t] \in \widehat{\mathbf{V}}_f$, we infer that π induces a bijection $\widehat{D}_g \to D_{fg}$, for any $g \in K[\mathbf{V}]$. Thus π is an open map, hence is a homeomorphism. (Note that this also reproves that π is continuous.) It remains to be shown that $\pi^*\colon \mathcal{O}_{D_f} \Rightarrow \mathcal{O}_{\widehat{\mathbf{V}}_f}$ induces isomorphisms on the level of K-algebras of functions:

By the sheaf properties it suffices to consider principal open subsets $D_{fg} \subseteq D_f$, where $g \in K[\mathbf{V}]$. Then we have $\pi^{-1}(D_{fg}) = \widehat{D}_g$, and

$$\pi^*_{D_{fg}}\colon \mathcal{O}_{D_f}(D_{fg}) = K[\mathbf{V}]_{fg} \to (K[\mathbf{V}]_f)_g = \mathcal{O}_{\widehat{\mathbf{V}}_f}(\widehat{D}_g)$$

boils down to the natural isomorphism $\pi^*_{fg}\colon K[\mathbf{V}]_{fg} \cong (K[\mathbf{V}]_f)_g$. ♯

In particular, we infer that the set $\Gamma(\mathcal{O}_{D_f}) = K[\mathbf{V}]_f$ of regular functions on D_f consists precisely of the morphisms of (affine) varieties $D_f \to L$. This entails that any quasi-affine variety $U \subseteq \mathbf{V}$ has an open covering consisting of affine open subsets. Thus by the sheaf properties we conclude that the set $\Gamma(\mathcal{O}_U)$ of regular functions on U consists precisely of the morphisms of varieties $U \to L$.

(23.2) Projective varieties. We keep the above notation, and let L be algebraically closed. We first consider $\mathbf{P} = \mathbf{P}^n(L)$, having structure sheaf $\mathcal{O}_{\mathbf{P}}$. For $i \in \{0, \ldots, n\}$ the principal open subset $D_i := D_{X_i} \subseteq \mathbf{P}$ has structure sheaf

$\mathcal{O}_{D_i} = \mathcal{O}_{\mathbf{P}}(D_i)$; recall that $\mathbf{P} = \bigcup_{i=0}^n D_i$. For notational simplicity we proceed to consider the case $i = 0$; the other open pieces are treated similarly.

Then homogenisation $\sigma\colon L^n \to D_0\colon v = [x_1, \ldots, x_n] \mapsto [1\colon x_1\colon \ldots\colon x_n] = v^\sharp$ and dehomogenisation $\tau\colon D_0 \to L^n\colon v = [x_0\colon \ldots\colon x_n] \mapsto [\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}] = v'$ are mutually inverse homeomorphisms. Still, for $g \in A^\sharp$ let $g' \in A$ be its dehomogenisation, and for $f \in A$ let $f^\sharp \in A^\sharp$ be its homogenisation. Recall that $\mathbf{A} := \mathbf{A}^n(L) = L^n$ is an affine variety, where $\Gamma(\mathcal{O}_{\mathbf{A}}) = A$.

Theorem. We have $\mathbf{A} \cong D_0$ as varieties.

Proof. Given $U \subseteq D_0$ open, we have to show $\sigma^*(\mathcal{O}_{\mathbf{P}}(U)) \subseteq \mathcal{O}_{\mathbf{A}}(\sigma^{-1}(U))$: Since the principal open subsets are a basis of the Zariski topology, by the sheaf properties it suffices to consider regular functions of shape $\varphi = \frac{f}{g}|_{D_g}$ on $D_g \subseteq D_0$, where $\frac{f}{g} \in \mathrm{Q}^\sharp(A^\sharp)$. For any $v \in \mathbf{A}$ we have $\sigma^*(g)(v) = g(v^\sharp) = g'(v)$. This implies $\sigma^*(g) = g'$ and $\sigma^{-1}(D_g) = D_{g'}$, thus $\sigma^*(\varphi) = \frac{f'}{g'}|_{D_{g'}}$, where $\frac{f'}{g'} \in \mathrm{Q}(A)$.

Conversely, given $V \subseteq \mathbf{A}$ open, we have to show $\tau^*(\mathcal{O}_{\mathbf{A}}(V)) \subseteq \mathcal{O}_{\mathbf{P}}(\tau^{-1}(V))$: Again, it suffices to consider regular functions of shape $\varphi = \frac{f}{g}|_{D_g}$, where $\frac{f}{g} \in \mathrm{Q}(A)$. For any $v \in D_0$ we have $\tau^*(g)(v) = g(v') = g(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0})(v)$. This implies $\tau^*(g) = \frac{g^\sharp}{X_0^{\deg(g)}}|_{D_0}$, and $\tau^{-1}(D_g) = D_0 \cap D_{g^\sharp} = D_{X_0 g^\sharp}$, thus we get $\tau^*(\varphi) = (X_0^{\deg(g)-\deg(f)} \cdot \frac{f^\sharp}{g^\sharp})|_{D_{X_0 g^\sharp}}$, where $X_0^{\deg(g)-\deg(f)} \cdot \frac{f^\sharp}{g^\sharp} \in \mathrm{Q}^\sharp(A^\sharp)$. ♯

Corollary. Let $\mathbf{V} \subseteq \mathbf{P}$ be closed and irreducible. Then we have $\Gamma(\mathcal{O}_{\mathbf{V}}) = K$.

Proof. Let $V_i := \mathbf{V} \cap D_i$, for $i \in \{0, \ldots, n\}$, so that $\mathbf{V} = \bigcup_{i=0}^n V_i$. If $V_n = \emptyset$, say, then we have $\mathbf{V} \subseteq \mathbf{P}^{n-1}$, and we may proceed with coordinates $X_0, \ldots, X_{n-1}$ instead. Thus we may assume that $V_i \neq \emptyset$ for all $i \in \{0, \ldots, n\}$. Moreover, if $n = 0$ then $\mathbf{V}$ is a singleton set; hence we may assume that $n \geq 1$.

Then $V_i \subseteq \mathbf{V}$ is open, hence dense, so that $\overline{V_i} = \mathbf{V}$. Since D_i is affine and $V_i \subseteq D_i$ is closed, we conclude that V_i is irreducible affine, such that $\mathcal{O}_{\mathbf{V}}(V_i) = \Gamma(\mathcal{O}_{V_i}) = K[V_i]$. Identifying $D_i \cong \mathbf{A}$ and $V_i \cong \mathbf{V}_i \subseteq \mathbf{A}$, we get $K[V_i] \cong K[\mathbf{V}_i] = K[\mathcal{X}^\sharp \setminus \{X_i\}]/\mathbf{I}_K(\mathbf{V}_i)$, where the vanishing ideals $\mathbf{I}_K^\sharp(\mathbf{V}) \trianglelefteq A^\sharp$ and $\mathbf{I}_K(\mathbf{V}_i)$ are related by (de)homogenisation at position i.

Now let $0 \neq \varphi \in \Gamma(\mathcal{O}_{\mathbf{V}})$. Then we have $\varphi|_{V_i} \in K[V_i]$, which is regular, thus continuous on the affine variety V_i. Hence from $\mathbf{V} = \bigcup_{i=0}^n V_i$ we conclude that φ is continuous, which since $V_i \subseteq \mathbf{V}$ is dense implies $\varphi|_{V_i} \neq 0$. Hence we have $\varphi|_{V_i} = f_i|_{V_i} = \frac{f_i}{X_i^{d_i}}|_{V_i}$, where $0 \neq \frac{f_i}{X_i^{d_i}} \in \mathrm{Q}^\sharp(K[\mathbf{V}])$ such that $f_i \in K[\mathbf{V}]$ is the homogenisation at position i of an element of $K[\mathbf{V}_i]$; thus we have $X_i \nmid f_i$ and $d_i = \deg(f_i)$. (Note that by the relation between the vanishing ideals involved this is well-defined indeed.)

For comparison, we consider $\emptyset \neq V_0 \cap V_i \subseteq V_0$, for $i \geq 1$, which is identified with the principal open subset $D_{X_i} \subseteq \mathbf{V}_0$. Dehomogenisation yields $(f_0)' = \frac{(f_i)'}{X_i^{d_i}} \in \mathcal{O}_{\mathbf{V}_0}(D_{X_i}) = K[\mathbf{V}_0]_{X_i} \subseteq K(\mathbf{V}_0)$. Thus we infer that $d_i = 0$, that is $f_i \in K$, for all $i \geq 1$. This implies $\varphi = f_0 = f_1 = \ldots = f_n \in K$, that is φ is constant. $\sharp$

Finally, we observe the following property of quasi-projective varieties:

Let $\mathbf{V} \subseteq \mathbf{P}$ be closed, and let $U \subseteq \mathbf{V}$ be open. We have see above that $D_i \cap \mathbf{V}$ is affine as well, so that $D_i \cap \mathbf{V} \subseteq \mathbf{V}$ is affine open, for all $i \in \{0, \ldots, n\}$. Thus, since $U \cap D_i = U \cap (D_i \cap \mathbf{V}) \subseteq D_i \cap \mathbf{V}$ is open, we conclude that $U \cap D_i$ is quasi-affine, hence $U \cap D_i \subseteq U$ is quasi-affine open. Thus $U = \bigcup_{i=0}^{n} (U \cap D_i)$ has a quasi-affine open covering, so that U also has an affine open covering.

In particular, by the sheaf properties we conclude that the set $\Gamma(\mathcal{O}_U)$ of regular functions on U consists precisely of the morphisms of varieties $U \to L$.
